Binomial Expansion I

$$(x+a)^n = a^n + \binom{n}{1} a^{n-1}x + \binom{n}{2} \mathrm{a}^{\mathrm{n}-2}\mathrm{x}^2 + \cdots + \binom{n}{n-1} \mathrm{ax}^{\mathrm{n}-1} + \mathrm{x}^{\mathrm{n}}$$

This equation is used when n is a particularly large number, and we only want the first few terms of the expansion. $\binom{n}{1}$ means ${}^nC_r$ which is a way of calculating how many different combinations of r items can you choose from a total n items.

For example, $(x+3)^5$ could be calculated by expanding this expression to $(x+3)\times(x+3)\times(x+3)\times(x+3)\times(x+3)$, and then multiplying each term by another. But using the binomial expansion, we can say that $(x+3)^5 = 3^5 + \binom{5}{1}3^4x + \binom{5}{2}3^3x^2 + \binom{5}{3}3^2x^3 + \binom{5}{4}3x^4 + x^5$
On all scientific calculators there should be an ${}^nC_r$ button, which can figure these calculations out for you.
$(x+3)^5 = 243 + 405x + 270x^2 + 90x^3 + 15x^4 + x^5$.

Proof

There is not exactly a proof for this formula *per se*, it merely requires some careful thought.

If we have the expansion $(x+a)^n$, it is clear that this will produce the following terms

a^n – as if you multiply every a term, there is only one way of doing this

${}^nC_1 \times a^{n-1}x$ – if you multiply every a term bar one (instead multiplying by an x term), then of the n brackets you must choose 1 to exclude, and there are ${}^nC_1$ ways of doing this

${}^nC_r \times a^{n-r} \times x^r$ – more generally, there are ${}^nC_r$ ways of choosing the r lots of x we wish to multiply by the remaining $n-r$ lots of a.

Using this logic, we now have the binomial expansion

$$(x+a)^n = a^n + \binom{n}{1} a^{n-1}x + \binom{n}{2} \mathrm{a}^{\mathrm{n}-2}\mathrm{x}^2 + \cdots + \binom{n}{n-1} \mathrm{ax}^{\mathrm{n}-1} + \mathrm{x}^{\mathrm{n}}$$

Note

You may have noticed from the pattern of the binomial expansion that it probably ought to be written

$$(x+a)^n = \binom{\boldsymbol{n}}{\boldsymbol{0}} a^n + \binom{n}{1} a^{n-1}x + \binom{n}{2} \mathrm{a}^{\mathrm{n}-2}\mathrm{x}^2 + \cdots + \binom{n}{n-1} \mathrm{ax}^{\mathrm{n}-1} + \binom{\boldsymbol{n}}{\boldsymbol{n}} \mathrm{x}^{\mathrm{n}}$$

But both $\binom{n}{0}$ and $\binom{n}{n}$ are equal to one, so we exclude them to avoid unnecessary clutter.

Instead of using ${}^nC_r$ buttons on your calculator, you can use Pascal's triangle. The row is the value of n and the column gives the value of r. Note how each term can be found by adding the term directly above it and the term above it and to the left. For example, ${}^5C_2$ is in bold, and the values of ${}^4C_1$ and ${}^4C_2$ are in italics as they can be added to find the value of ${}^5C_2$

n/r	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	*4*	*6*	4	1		
5	1	5	**10**	10	5	1	
6	1	6	15	20	15	6	1

See also
- Binomial Expansion II
- Binomial Distribution

References
Turner, L. K. (1976). *Advanced Mathematics*. London: Longman. pp.233-234.